arXiv:0705.2874v1 [math.AT] 21 May 2007

Combinatorial Morse theory and minimality of hyperplane arrangements

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January 2007

1 Introduction

In [DP03], [Ra02] it was proven that the complement to a hyperplane arrangement in $\mathbb{C}^n$ is a *minimal* space, i.e. it has the homotopy type of a CW-complex with exactly as many i-cells as the i-th Betti number b_i. The arguments use (relative) Morse theory and Lefschetz type theorems.

This result of "existence" was refined in the case of complexified real arrangements in [Yo05]. The author consider a flag $V_0 \subset V_1 \subset \cdots \subset V_n \subset \mathbb{R}^n$, $dim(V_i) = i$, which is *generic* with respect to the arrangement, i.e. V_i intersects transversally all codimensional$-i$ intersections of hyperplanes. The interesting main result is a correspondence between the $k-$cells of the minimal complex and the set of *chambers* which intersect V_k but do not intersect V_{k-1}. The arguments still use the Morse theoretic proof of Lefschetz theorem, and some analysis of the critical cells is given. Unfortunately, the description does not allow to understand exactly the attaching maps of the cells of a minimal complex.

In this paper we give, for a complexified real arrangement $\mathcal{A}$, an explicit Lefschetz theorem - free description of a minimal CW-complex. The idea is that, since an explicit $CW-$complex $\mathbf{S}$ which describes the homotopy type of the complement already exists (see [Sal87]), even if not minimal, one can work over such complex trying to "minimize" it. A natural tool for doing that is to use *combinatorial Morse theory* over $\mathbf{S}$. We follow the approach of [Fo98], [Fo02] to combinatorial Morse theory (i.e., Morse theory over CW-complexes).

So, we explicitly construct a combinatorial gradient vector field over $\mathbf{S}$, related to a given system of polar coordinates in $\mathbb{R}^n$ which is *generic* with respect to the arrangement $\mathcal{A}$. Let $\mathcal{S}$ be the set of all *facets* of the stratification of $\mathbb{R}^n$ induced by the arrangement $\mathcal{A}$ (see [Bou68]). Then $\mathcal{S}$ has a natural partial ordering given by $F \prec G$ iff $clos(F) \supset G$. Our definition of genericity of a coordinate system, which is stronger than that used in [Yo05], allows to give a *total ordering* $\lhd$ on $\mathcal{S}$, which we call the *polar ordering* of the facets.

The k-cells in $\mathbf{S}$ are in one-to-one correspondence with the pairs $[C \prec F^k]$, where C is a chamber in $\mathcal{S}$ and F^k is a codimensional-k facet of $\mathcal{S}$ which is contained in the closure of C. Then the gradient field can be recursively defined

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as the set of pairs

$$([C \prec F^{k-1}], [C \prec F^k])$$

such that $F^{k-1} \prec F^k$ and $F^k \lhd F^{k-1}$, and such that the origin cell of the pair is not the end cell of another pair of the field. We also give a non-recursive equivalent characterization of the field (thm. 6, (ii)) only in terms of the partial ordering $\prec$ and of the total ordering $\lhd$.

Analog to index$-k$ critical points in the standard Morse theory, there are *singular cells* of dimension k: they are those k-cells which do not belong to the gradient field (see [Fo98]). In our situation, they are given (see corollary 4.8) in terms of the orderings by those $[C \prec F^k]$ such that

$$\begin{array}{llllll} i) & F^k \lhd F^{k+1}, & \forall & F^{k+1} & s.t. & F^k \prec F^{k+1} \\ ii) & F^{k-1} \lhd F^k, & \forall & F^{k-1} & s.t. & C \prec F^{k-1} \prec F^k \end{array}$$

It is easy to see that associating to a singular cell $[C \prec F^k]$ the unique chamber C' which is *opposite* to C with respect to F^k, gives a one-to-one correspondence between the set of singular cells in $\mathbf{S}$ and the set of all chambers in $\mathcal{S}$. So we also derive by our method the main result in [Yo05].

Minimality property of the complement follows easily, so the above singular cells of $\mathbf{S}$ give an explicit basis for the integral cohomology, which depends on the system of polar coordinates (we call such a basis a *polar basis* for the cohomology). A minimal complex is obtained from $\mathbf{S}$ by contracting all pairs of cells which belong to the vector field.

Our construction gives also an explicit algebraic complex which computes local system cohomology of $M(\mathcal{A})$. In dimension k such complex has one generator for each singular cell of $\mathbf{S}$. The boundary operator is obtained by a method which is the combinatorial analog to "integrating over all paths" which satisfy some conditions. We give a reduced formula for the boundary, which is effectively computable in terms only of the two orderings $\prec$, $\lhd$. For abelian local systems, the boundary operator assumes an even nicer reduced form. There exists a vast literature about calculation of local system cohomology on the complement to an arrangement: several people constructed algebraic complexes computing local coefficient cohomology, in the abelian case (see for example [Co93, CO00, ESV92, Ko86, LY00, STV95, Su02, Yo05]). Our method seems to be more effective than the previous ones.

In the last part we find a generic polar ordering on the braid arrangement. We give a description of the complex $\mathbf{S}$ in this case in terms of *tableaux* of a special kind; next, we characterize the *singular* tableaux and we find an algorithm to compare two tableaux with respect to the polar ordering.

Some of the most immediate remaining problems are: first, compare polar bases with the well-known $nbc-$bases of the cohomology (see [BZ92, OT92]); second, characterize polar orderings in a purely combinatorial way (so, using an *oriented matroid* counterpart of generic polar coordinates). [1]

2 n-dimensional polar coordinates

For reader's convenience, we recall here n-dimensional polar coordinates. Since usually one knows only standard 3-dimensional formulas, we give here coordinate changes in general.

[1] A preliminary version of this paper was published as a preprint in [SS07]

Start with an orthonormal basis

$$\mathbf{e}_1, ..., \mathbf{e}_n$$

of the Euclidean $n-$dimensional space V and let

$$P \ \equiv \ (x_1, ..., x_n)$$

the associated cartesian coordinates of a point P. We will confuse the point P and the vector OP, O being the origin of the coordinate system.

Let in general

$$pr_W : \ V \ \rightarrow \ W$$

be the orthogonal projection onto a subspace W of V. Consider the two flags of subspaces

$$V_i \ = < \mathbf{e}_1, ..., \mathbf{e}_i >, \quad i = 0, ..., n \ (V_0 = 0)$$

and

$$W_i \ = < \mathbf{e}_i, ..., \mathbf{e}_n >, \quad i = 1, ..., n.$$

Let

$$P_i \ := \ pr_{W_i}(P), \ i = 1, ..., n$$

(so $P_1 = P$). One has

$$P_i \ = \ pr_{W_i}(P_j), \ j \leq i$$

so there are orthogonal decompositions

$$P_i \ = \ P_{i+1} \ + \ x_i \mathbf{e}_i, \ x_i \in \mathbb{R}, \ i = 1, ..., n \tag{1}$$

(set $P_{n+1} = 0$)

Clearly

$$\begin{array}{rcl} P_i \ = \ 0 & \Rightarrow & P_j = 0 \text{ for } j \geq i \\ P_i \ \neq 0 & \Rightarrow & P_j \neq 0 \text{ for } j \leq i \end{array} \tag{2}$$

Let

$$\theta_{n-1} \in (-\pi, \pi]$$

be the angle that OP_{n-1} forms with e_{n-1} (in the 2-plane W_{n-1}). Let then

$$\theta_i \in [0, \pi], \quad i = 1, ..., n-2$$

be the angle that OP_i makes with e_i.

The polar coordinates of P will be given by the "module"

$$\rho \ = \ \|P\|$$

more (if $P \neq 0$) "arguments"

$$\theta_1, ..., \theta_{n-1}$$

(defined only for $i \leq max\{j : \ P_j \neq 0\}$).

The coordinate change between polar and cartesian coordinates is given by

$$
\begin{array}{rcl}
x_1 & = & \rho\cos(\theta_1) \\
x_2 & = & \rho\sin(\theta_1)\cos(\theta_2) \\
\vdots & \vdots & \vdots \\
x_i & = & \rho\sin(\theta_1)...\sin(\theta_{i-1})\cos(\theta_i) \\
\vdots & \vdots & \vdots \\
x_{n-1} & = & \rho\sin(\theta_1)...\sin(\theta_{n-2})\cos(\theta_{n-1}) \\
x_n & = & \rho\sin(\theta_1)...\sin(\theta_{n-1})
\end{array} \tag{3}
$$

Notice that these formulas make sense always if we conventionally set $\theta_i = 0$ for $P_i = 0$.

The inverse formulas are

$$
\begin{array}{rcl}
\rho^2 & = & x_1^2 + ... + x_n^2 \\
\cos^2(\theta_1) & = & \frac{x_1^2}{x_1^2+...+x_n^2} \\
\vdots & \vdots & \vdots \\
\cos^2(\theta_i) & = & \frac{x_i^2}{x_i^2+...+x_n^2} \\
\vdots & \vdots & \vdots \\
\cos^2(\theta_{n-1}) & = & \frac{x_{n-1}^2}{x_{n-1}^2+x_n^2}
\end{array} \tag{4}
$$

3 Combinatorial Morse theory

We recall here the main points of Morse theory for CW-complexes, from a combinatorial viewpoint. All the definitions and results in this section are taken from [Fo98], [Fo02].

We restrict to the case of our interest, that of *regular* $CW-$complexes.

3.1 Discrete Morse functions

Let M be a finite regular CW-complex, let K denote the set of cells of M, partially ordered by

$$\sigma \ < \ \tau \quad \Leftrightarrow \quad \sigma \subset \tau,$$

and K_p the cells of dimension p.

Definition 3.1 *A discrete Morse function on M is a function*

$$f : K \longrightarrow \mathbb{R}$$

satisfying for all $\sigma^{(p)} \in K_p$ the two conditions

$$
\begin{array}{lll}
(i) & \sharp\{\tau^{(p+1)} > \sigma^{(p)} \mid f(\tau^{(p+1)}) \leq f(\sigma^{(p)})\} & \leq \ 1 \\
(ii) & \sharp\{v^{(p-1)} < \sigma^{(p)} \mid f(\sigma^{(p)}) \leq f(v^{(p-1)})\} & \leq \ 1
\end{array}
$$

We say that $\sigma^{(p)} \in K_p$ is a critical cell *of index p if the cardinality of both these sets is* 0.

Remark 3.2 *One can show that, for any given cell of M, at least one of the two cardinalities in (i), (ii) is* 0 *([Fo98]).*

Let $m_p(f)$ denote the number of critical cells of f of index p. As in the standard theory one can show (see [Fo98])

Proposition 3.3 *M is homotopy equivalent to a CW-complex with exactly $m_p(f)$ cells of dimension p.*

3.2 Gradient vector fields

Let f be a discrete Morse function on a CW-complex M. One can define the discrete gradient vector field V_f of f as:

$$V_f \;=\; \{(\sigma^{(p)}, \tau^{(p+1)}) | f(\tau^{(p+1)}) \leq f(\sigma^{(p)})\}.$$

By definition of Morse function, each cell belongs to at most one pair of V_f. More generally, one defines

Definition 3.4 *A discrete vector field V on M is a collection of pairs $(\sigma^{(p)}, \tau^{(p+1)})$ of cells such that each cell of M belongs to at most one pair of V.*

Given a discrete vector field V on M, a V-path is a sequence of cells

$$\sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \tau_1^{(p+1)}, \sigma_2^{(p)}, \cdots, \tau_r^{(p+1)}, \sigma_{r+1}^{(p)} \tag{1}$$

such that for each $i = 0, \cdots, r$ $(\sigma_r^{(p)}, \tau_r^{(p+1)}) \in V$ and $\sigma_i^{(p)} \neq \sigma_{(i+1)}^{p} < \tau_i^{(p+1)}$. Such a path is a non trivial closed path if $0 \leq r$ and $\sigma_0^{(p)} = \sigma_{(r+1)}^{p}$. One has:

Theorem 1 *A discrete vector field V is the gradient vector field of a discrete Morse function if and only if there are no non-trivial closed V-path.*

Remark 3.5 *An equivalent combinatorial definition of discrete vector field is that of* matching *over the Hasse diagram of the poset associated to the CW-complex (see for ex. [Fo02]).*

4 Applications to Hyperplane arrangements

4.1 Notations and recalls

Let $\mathcal{A} \;=\; \{H\}$ be a finite affine hyperplane arrangement in $\mathbb{R}^n$. Assume $\mathcal{A}$ essential, so that the minimal dimensional non-empty intersections of hyperplanes are points (which we call *vertices* of the arrangement). Equivalently, the maximal elements of the associated *intersection lattice* $L(\mathcal{A})$ (see [OT92]) have rank n.

Let

$$M(\mathcal{A}) \;=\; \mathbb{C}^n \setminus \bigcup_{H \in A} H_{\mathbb{C}}$$

be the complement to the complexified arrangement. We use the regular CW-complex $\mathbf{S} = \mathbf{S}(\mathcal{A})$ constructed in [Sal87] which is a deformation retract of $M(\mathcal{A})$ (see also [GR89], [BZ92], [OT92], [Sal94]). Here we recall very briefly some notations and properties.

Let
$$\mathcal{S} := \{F^k\}$$
be the stratification of $\mathbb{R}^n$ into facets F^k which is induced by the arrangement (see [Bou68]), where exponent k stands for codimension. Then $\mathcal{S}$ has standard partial ordering
$$F^i \prec F^j \quad \text{iff} \quad clos(F^i) \supset F^j$$
Recall that k-cells of $\mathbf{S}$ bijectively correspond to pairs
$$[C \prec F^k]$$
where $C = F^0$ is a chamber of $\mathcal{S}$.

Let $|F|$ be the affine subspace spanned by F, and let us consider the subarrangement
$$\mathcal{A}_F \;=\; \{H \in \mathcal{A} \;:\; F \subset H\}.$$
A cell $[C \prec F^k]$ is in the boundary of $[D \prec G^j]$ $(k < j)$ iff

i) $F^k \prec G^j$
ii) the chambers C and D are contained in the same chamber of $\mathcal{A}_{F^k}$.

Previous conditions are equivalent to say that C is the chamber of $\mathcal{A}$ which is "closest" to D among those which contain F^k in their closure.

Notation 4.1 *i) We denote the chamber D which appear in the boundary cell $[D \prec G^j]$ of a cell $[C \prec F^k]$ by $C.G^j$.*

ii) More generally, given a chamber C and a facet F, we denote by $C.F$ the unique chamber containing F and lying in the same chamber as C in $\mathcal{A}_{F^k}$. Given two facets F, G we will use also for $(C.F).G$ the notation (without brackets) $C.F.G$.

It is possible to realize $\mathbf{S}$ inside $\mathbb{C}^n$ with explicitly given attaching maps of the cells (see [Sal87]). Recall also that the construction can be given for any *oriented matroid* (see the above cited references).

4.2 Generic polar coordinates

In general, we distinguish between *bounded* and *unbounded* facets. Let $B(\mathcal{S})$ be the union of bounded facets in $\mathcal{S}$. When $\mathcal{A}$ is central and essential (i.e $\cap_{H\in\mathcal{A}} H$ is a single point $O \in V$) then $B(\mathcal{S}) = \{O\}$. In general, it is known that $B(\mathcal{S})$ is a compact connected subset of V and the closure of a small open neighborhood U of $B(\mathcal{S})$ is homeomorphic to a ball (so U is an open ball; see for ex. [Sal87]).

Given a system of polar coordinates associated to $O, \mathbf{e}_1, ..., \mathbf{e}_n$, the coordinate subspace V_i , $i = 1, ..., n$ (see section 1) is divided by V_{i-1} into two components:
$$V_i \setminus V_{i-1} \;=\; V_i(0) \cup V_i(\pi)$$
where
$$V_i(0) \;=\; \{P : \;\theta_i(P) = 0\}$$

and

$$V_i(\pi) \ = \ \{P: \ \theta_i(P) = \pi\}$$

More generally, we indicate by

$$V_i(\bar{\theta}_i, ..., \bar{\theta}_{n-1}) \ := \ \{P: \ \theta_i(P) = \bar{\theta}_i, ..., \theta_{n-1}(P) = \bar{\theta}_{n-1}\} \tag{5}$$

where by convention $\bar{\theta}_j = 0$ or $\pi \ \Rightarrow \ \bar{\theta}_k = 0$ for all $k > j$; so in particular, $V_i(0) = V_i(0, ..., 0)$ and $V_i(\pi) = V_i(\pi, 0, ..., 0)$ ($n-i$ components). The space $V_i(\bar{\theta})$ is an $i-$dimensional open half-subspace in the euclidean space V, and we denote by $|V_i(\bar{\theta})|$ the subspace which is spanned by it. We have from (3)

$$|V_i(\bar{\theta}_i, ..., \bar{\theta}_{n-1})| \ = < \mathbf{e}_1, ..., \mathbf{e}_{i-1}, \bar{\mathbf{e}} >$$

where

$$\bar{\mathbf{e}} = \bar{\mathbf{e}}(\theta_i, ..., \theta_{n-1}) := \sum_{j=i}^{n} \left(\prod_{k=i}^{j-1} \sin(\theta_k) \right) \cos(\theta_j) \ \mathbf{e}_j$$

For all $\delta \in (0, \pi/2)$ the space

$$\tilde{B} := \ \tilde{B}(\delta) := \ \{P: \ \theta_i(P) \in (0, \delta), \ i = 1, ..., n-1, \ \rho(P) > 0\}$$

is an open cone contained in $\mathbb{R}^n_+$.

Definition 4.2 *We say that a system of polar coordinates in $\mathbb{R}^n$, defined by an origin O and a base $\mathbf{e}_1, ..., \mathbf{e}_n$, is* generic *with respect to the arrangement $\mathcal{A}$ if it satisfies the following conditions:*

i) the origin O is contained in a chamber C_0 of $\mathcal{A}$;
ii) there exist $\delta \in (0, \pi/2)$ such that

$$B(\mathcal{S}) \ \subset \ \tilde{B} = \tilde{B}(\delta);$$

(therefore, for each facet $F \in \mathcal{S}$ one has $F \cap \tilde{B} \neq \emptyset$);
iii) subspaces $V_i(\bar{\theta}) = V_i(\bar{\theta}_i, ..., \bar{\theta}_{n-1})$ which intersect $clos(\tilde{B})$ (so $\bar{\theta}_j \in [0, \delta]$ for $j = i, ..., n-1$) are generic with respect to $\mathcal{A}$, in the sense that, for each codim$-k$ subspace $L \in L(\mathcal{A})$,

$$i \geq k \ \Rightarrow \ V_i(\bar{\theta}) \cap L \cap clos(\tilde{B}) \neq \emptyset \textit{ and } dim(|V_i(\bar{\theta})| \cap L) \ = \ i - k.$$

It is easy to see that genericity condition implies that the origin O of coordinates belongs to an unbounded chamber. It turns out that such chamber must intersects the infinity hyperplane H_∞ into a relatively open set. This is equivalent to say that the sub-arrangement given by the walls of the chamber is essential.

In fact we have

Theorem 2 *For each unbounded chamber C such that $C \cap H_\infty$ is relatively open, the set of points $O \in C$ such that there exists a polar coordinate system centered in O and generic with respect to $\mathcal{A}$ forms an open subset of C.*

Proof. We proceed by first proving the following

Lemma 4.3 *Let $\mathcal{A}$ be a central essential arrangement in V. Then there exist orthonormal frames $\boldsymbol{e}_1, ..., \boldsymbol{e}_n$ which are* generic *with respect to $\mathcal{A}$, in the sense that each subspace $V_i := < \boldsymbol{e}_1, ..., \boldsymbol{e}_i >, \ i = 1, ..., n$, intersects transversally each $L \in L(\mathcal{A})$. Given a chamber C, the first vector $\boldsymbol{e}_1$ can be any vector inside C.*

Actually, the set of generic frames is open inside the space of orthonormal frames in V.

Proof of lemma. Let O' be the intersection of all hyperplanes, and take an orthonormal coordinate system with basis $\mathbf{e}'_1, ..., \mathbf{e}'_n$. Then each hyperplane H is given by a linear form

$$H_i \ = \ \{x : (\alpha_i \ \cdot \ x) \ = \ 0\}, \ i = 1, ..., |\mathcal{A}|$$

where we denote by $(\ \cdot \ )$ the canonical inner product. Any codimensional$-k$ L is given by an intersection of k linearly independent hyperplanes $H_{i_1}, ..., H_{i_k}$ of $\mathcal{A}$. The genericity condition on a frame $\mathbf{e}_1, ..., \mathbf{e}_n$ is written as

$$rk \left[(\alpha_{i_r} \ \cdot \ \mathbf{e}_s) \right]_{\substack{r=1,...,k \\ s=1,...,i}} \ = \ min\{k, i\}$$

or, equivalently

$$rk \left[(\alpha_{i_r} \ \cdot \ \mathbf{e}_s) \right]_{\substack{r=1,...,k \\ s=1,...,k}} \ = \ k. \tag{7}$$

It is clear that genericity applied to V_1 gives that $\mathbf{e}_1$ is not contained in any hyperplane, i.e. it belongs to some chamber of $\mathcal{A}$. Equation (7) is easily translated into the equivalent one

$$dim(< \alpha_{i_1}, ..., \alpha_{i_k}, \mathbf{e}_{k+1}, ..., \mathbf{e}_n >) = n. \tag{8}$$

Passing to the dual space V^* by using the inner product, the set of all hyperplanes

$$< \alpha_{i_1}, ..., \alpha_{i_{n-1}} > \subset V^*$$

gives an arrangement $\mathcal{A}^*$. Since $\mathbf{e}_1$ belongs to a chamber of $\mathcal{A}$, each subspace $< \alpha_{i_1}, ..., \alpha_{i_k} >$ intersects transversally the orthogonal $\mathbf{e}_1^{\perp}$, so $\mathcal{A}^*$ induces an arrangement $\mathcal{A}_1 := \mathcal{A}^* \cap \mathbf{e}_1^{\perp}$ over $\mathbf{e}_1^{\perp}$. Condition (8) requires an orthonormal basis $\mathbf{e}_2, ..., \mathbf{e}_n \in \mathbf{e}_1^{\perp}$ which is generic with respect to the flag

$$V'_1 := < \mathbf{e}_n >, ..., V'_i := < \mathbf{e}_{n-i+1}, ..., \mathbf{e}_n >, ...$$

Then we conclude the proof of the first and second assertions by induction on n.

For the last one, notice that

$$\mathbf{e}_1, \ \mathbf{e}_n, \ \mathbf{e}_2, \ \mathbf{e}_{n-1}, \ ...$$

can vary respectively in a chamber of the arrangements

$$\mathcal{A}, \ \mathcal{A}_1 = \mathcal{A}^* \cap \mathbf{e}_1^{\perp}, \ \mathcal{A}_{1,n} := (\mathcal{A}_1)^* \cap \mathbf{e}_n^{\perp}, \ \mathcal{A}_{1,n,2} := \mathcal{A}_{1,n}^* \cap \mathbf{e}_2^{\perp}, \ ...$$

which is an open set inside orthonormal frames. $\square$

We come back to the prove of theorem.

Case I: $\mathcal{A}$ central and essential.
Let O' be the center of $\mathcal{A}$. According to the previous lemma we can find $\mathbf{e}_1, ..., \mathbf{e}_n$ generic with respect to $\mathcal{A}$, and with $\mathbf{e}_1 := \frac{\vec{OO'}}{\|OO'\|}$. If we consider a system of polar coordinates associated to $O, \mathbf{e}_1, ..., \mathbf{e}_n$ then the subspaces V_i satisfy condition (iii) of genericity. Perform a small translation onto $\mathcal{A}$,

$$x_i \ \rightarrow \ x_i + \sigma, \ 0 < \sigma << 1$$

which moves the center O' into the positive octant. Then if

$$\sigma << \delta << 1$$

all conditions in definition 3.1 are satisfied by continuity and the fact that genericity is an open condition.

General case.
In case of an affine arrangement referred to a system of cartesian coordinates $O', \mathbf{e}'_1, ..., \mathbf{e}'_n$, hyperplanes are written as

$$H_i = \{x : \ (\alpha_i \ \cdot \ x) \ = \ a_i\} \ , i = 1, ..., |\mathcal{A}|.$$

Let

$$H_i^0 := \{(\alpha_i \ \cdot \ x) \ = \ 0)\}$$

be the *direction* of H_i and let $\mathcal{A}_0$ be the associated central arrangement. Notice that if $\mathcal{A}$ is essential then so is $\mathcal{A}_0$. We can assume without loss of generality that $\|\alpha_i\| = 1, \ \forall i$, so the vector

$$a_i \cdot \alpha_i$$

represents the translation taking H_i^0 into H_i.

Let C be an unbounded chamber of $\mathcal{A}$ such that $C \cap H_\infty$ is relatively open in H_∞. Then the directions of the walls of C are the walls of a chamber C' in $\mathcal{A}_0$. By previous case, there exist points O in C' and systems of polar coordinates $O, \mathbf{e}_1, ..., \mathbf{e}_n$ which are generic with respect to $\mathcal{A}_0$. Let $\delta > 0$ satisfy definition 4.2 for one of such systems. We can assume (up to a homotethy of center O')

$$|a_i| << \delta, \ \forall i.$$

Then the same system satisfies the definition for $\mathcal{A}$.

Of course, the condition of genericity is open, so we finish the proof of the theorem. □

4.3 Orderings on $\mathcal{S}$

Fix a system of generic polar coordinates, associated to a center O and frame $\mathbf{e}_1, ..., \mathbf{e}_n$. Let $\delta > 0$ be the number coming from definition 3.1. We denote for brevity $\bar{B} := clos(\tilde{B}(\delta))$. Each point P has polar coordinates $P \equiv (\theta_0, \theta_1, ..., \theta_{n-1})$, where we use the convention $\theta_0 := \rho$.

We remark that when the pole O is very far, the polar coordinates of one point inside $\tilde{B}(\delta)$ are approximately the same as its standard Cartesian coordinates.

Notice that (5) makes sense also for $i = 0$, being

$$V_0(\bar{\theta}_0, \bar{\theta}_1, ..., \bar{\theta}_{n-1})$$

given by a single point P with

$$\rho(P) = \bar{\theta}_0,\ \theta_1(P) = \bar{\theta}_1,\ \dots\ ,\ \theta_{n-1}(P) = \bar{\theta}_{n-1}.$$

Given a codimensional$-k$ facet $F \in \mathcal{S}$, let us denote by

$$F(\theta) := F(\theta_i, ..., \theta_{n-1}) := F \cap V_i(\theta_i, ..., \theta_{n-1}), \quad \theta_j \in [0, \delta],\ j = i, ..., n-1$$

(notice: $F = F(\theta) = F \cap V_n$ with $\theta = \emptyset$.)

By genericity conditions, if $i \geq k$ then $F(\theta)$ is either empty or it is a codimensional $k + n - i$ facet contained in $V_i(\theta_i, ..., \theta_{n-1})$.

Let us set, for every facet $F(\theta)$,

$$i_{F(\theta)} \ := \ min\{j \geq 0 :\ V_j \cap clos(F(\theta)) \neq \emptyset\}.$$

Still by genericity, setting $L := |F(\theta)|$, one has

$$L \cap V_j \ \neq \ \emptyset \ \Leftrightarrow \ j \geq codim(F(\theta))$$

so also

$$i_{F(\theta)} \ \geq \ codim(F(\theta)) \qquad (9).$$

When the facet $F(\theta) := F(\theta_i, ..., \theta_{n-1}),\ i > 0$, is not empty and $i_{F(\theta)} \geq i$ (i.e., $clos(F(\theta)) \cap V_{i-1} = \emptyset$), then among its vertices ($0-$dimensional facets in its boundary) there exists, still by genericity, a unique one

$$P := P_{F(\theta)} \in clos(F(\theta)) \qquad (10)$$

such that

$$\theta_{i-1}(P) \ = \ min\{\theta_{i-1}(Q) :\ Q \in clos(F(\theta))\} \qquad (11)$$

(of course, $P_{F(\theta)} = F(\theta)$ if $dim(F(\theta)) = 0$, i.e. $i = k$).

When $i_{F(\theta)} < i$ then the point P of (10) is either the origin 0 ($\Leftrightarrow\ i_{F(\theta)} = 0$ $\Leftrightarrow$ F is the base chamber C_0) or it is the unique one such that

$$\theta_{i_{F(\theta)}-1}(P) \ = \ min\{\theta_{i_{F(\theta)}-1}(Q) :\ Q \in clos(F(\theta)) \cap V_{i_{F(\theta)}}\} \qquad (12)$$

Definition 4.4 *Given any facet $F(\theta) = F(\theta_i, ..., \theta_{n-1})$ let us denote by*

$$P_{F(\theta)} \in clos(F(\theta))$$

the "minimum" vertex of $clos(F(\theta)) \cap\ V_{i_{F(\theta)}}$ (as in (10))
(for $F \in \mathcal{S}$ we briefly write P_F).

We associate to the facet $F(\theta)$ the $n-$vector of polar coordinates of $P_{F(\theta)}$

$$\Theta(F(\theta)) \ := \ (\theta_0(F(\theta)), ..., \theta_{i_{F(\theta)}-1}(F(\theta)), 0, ..., 0)$$

($n - i_{F(\theta)}$ zeroes) where we set

$$\theta_j(F(\theta)) := \theta_j(P_{F(\theta)}), \quad j = 0, ..., i_F - 1.$$

Notice that when $i_{F(\theta)} \geq i$ then all coordinates θ_j of θ with $j \geq i_{F(\theta)}$ must be zero.

We want to define another ordering over the poset $\mathcal{S}, \prec$. We give a recursive definition, actually ordering all facets in $V_i(\theta)$ for any given $\theta = (\theta_1, ..., \theta_{n-1})$.

Definition 4.5 (Polar Ordering) *Given $F,\ G \in \mathcal{S}$, and given $\bar{\theta} = (\bar{\theta}_i, ..., \bar{\theta}_{n-1})$, $0 \leq i \leq n$, $\bar{\theta}_j \in [0, \delta]$ for $j \in i, .., n-1$, ($\bar{\theta} = \emptyset$ for $i = n$) such that $F(\bar{\theta}), G(\bar{\theta}) \neq \emptyset$, we set*

$$F(\bar{\theta}) \ \lhd \ G(\bar{\theta})$$

iff one of the following cases occur:

i) $P_{F(\bar{\theta})} \neq P_{G(\bar{\theta})}$. Then $\Theta(F(\bar{\theta})) \ < \ \Theta(G(\bar{\theta}))$ according to the anti-lexicographic ordering of the coordinates (i.e., the lexicographic ordering starting from the last coordinate).

ii) $P_{F(\bar{\theta})} = P_{G(\bar{\theta})}$. Then either

iia) $dim(F(\bar{\theta})) = 0$ (so $P_{F(\bar{\theta})} = F(\bar{\theta})$) and $F(\bar{\theta}) \neq G(\bar{\theta})$ (so $dim(G(\bar{\theta})) > 0$)

or

iib) $dim(F(\bar{\theta})) > 0,\ dim(G(\bar{\theta})) > 0$. In this case let $i_0 := i_{F(\bar{\theta})} = i_{G(\bar{\theta})}$.
When $i_0 \geq i$ (case (11)) one can write

$$\Theta(F(\bar{\theta})) = \Theta(G(\bar{\theta})) = (\tilde{\theta}_0, ..., \tilde{\theta}_{i-1}, \bar{\theta}_i, ..., \bar{\theta}_{i_0-1}, 0, ..., 0).$$

Then $\forall \epsilon,\ 0 < \epsilon << \delta$, it must happen

$$F(\tilde{\theta}_{i-1} + \epsilon, \bar{\theta}_i, ..., \bar{\theta}_{i_0-1}, 0, ..., 0) \ \lhd \ G(\tilde{\theta}_{i-1} + \epsilon, \bar{\theta}_i, ..., \bar{\theta}_{i_0-1}, 0, ..., 0).$$

If $i_0 < i$ (as in (12)) then one can write

$$\Theta(F(\bar{\theta})) = \Theta(G(\bar{\theta})) = (\tilde{\theta}_0, ..., \tilde{\theta}_{i_0-1}, 0, ..., 0).$$

Then $\forall \epsilon,\ 0 < \epsilon << \delta$, it must happen

$$F(\tilde{\theta}_{i_0-1} + \epsilon, 0, ..., 0) \ \lhd \ G(\tilde{\theta}_{i_0-1} + \epsilon, 0, ..., 0).$$

($n - i_0$ zeroes) □

Condition (iib) says that one has to move a little bit the suitable $V_j(\theta')$ which intersects $clos(F(\theta))$ and $clos(G(\theta))$ in a point $P(F(\theta)) = P(G(\theta))$ (according to (11) or (12)), and consider the facets which are obtained by intersection with this "moved" subspace.

It is quit clear from the definition that irriflessivity and transitivity hold for $\lhd$ so we have

Theorem 3 *Polar ordering $\lhd$ is a total ordering on the facets of $V_i(\bar{\theta})$, for any given $\bar{\theta} = (\bar{\theta}_i, ..., \bar{\theta}_{n-1})$. In particular (taking $\bar{\theta} = \emptyset$) it gives a total ordering on $\mathcal{S}$.* □

The following property, comparing polar ordering with the partial ordering $\prec$, will be very useful.

Theorem 4 *Each codimensional-k facet $F^k \in \mathcal{S}$ $(k < n)$ such that $F^k \cap V_k = \emptyset$ has the following property: among all codimensional-$(k+1)$ facets G^{k+1} with $F^k \prec G^{k+1}$, there exists a unique one F^{k+1} such that*

$$F^{k+1} \lhd F^k.$$

If $F^k \cap V_k \neq \emptyset$ (so $F^k \cap V_k = P(F^k)$) then

$$F^k \lhd G^{k+1}, \quad \forall G^{k+1} \text{ with } F^k \prec G^{k+1}.$$

Proof. In the latter case, where $F^k \cap V_k = P(F^k)$, for every facet G^{k+1} in the closure of F^k one has $P(G^{k+1}) \notin clos(V_k)$ (by (9)), so $F^k \lhd G^{k+1}$.

In general, for all facets G contained in the closure of F^k, one has either $P(G) \neq P(F^k)$ and $\Theta(F^k) < \Theta(G)$, so $F^k \lhd G$, or $P(G) = P(F^k)$. For those G^{k+1} such that $P(G^{k+1}) = P(F^k)$ one reduces, after $\epsilon-$deforming (may be several times) like in definition 4.5, to the case where F is a one-dimensional facet contained in some $V_h \setminus V_{h-1}$, with $h \geq 1$, and for such case the assertion is clear. $\square$

Let $\mathcal{S}^{(k)} := \mathcal{S} \ \cap \ V_k$ be the stratification induced onto the coordinate subspace V_k. A codimensional-j facet in V_k is the intersection with V_k of a unique codimensional-j facet in $\mathcal{S}$, $j \leq k$. Let $\lhd_k$ be the polar ordering of $\mathcal{S}^{(k)}$, induced by the polar coordinates associated to the basis $\mathbf{e}_1, ..., \mathbf{e}_k$ of V_k. By construction, for all $F, \ G \in \mathcal{S}$ which intersect V_k, one has

$$F \cap V_k \ \lhd_k \ G \cap V_k \quad \text{iff} \quad F \lhd G.$$

So we can say that $\lhd_k$ is the *restriction* of $\lhd$ to V_k and also $\lhd_k$ is the restriction of $\lhd_h$ for $k < h$.

By genericity conditions, for each $F^k \in \mathcal{S}$ there exists a unique F_0^k with the same support and intersecting V_k (in one point).

The following recursive characterization of the polar ordering will be used later. The proof is a direct consequence of definition 4.5 and theorem 4.

Theorem 5 *Assume that, for all $k = 0, ..., n$, we know the polar ordering of all the 0-facets (=codimensional-k facets) of $\mathcal{S}^{(k)}$ (in particular, $\forall$ F^k we know whether $F^k \cap V_k \neq \emptyset$). Then we can reconstruct the polar ordering of all $\mathcal{S}$. Assuming we know it for all facets of codimension $\geq k+1$, then given $F^k, \ G^k$ we have:*

- if both $F^k, \ G^k$ intersect V_k then the ordering is the same as the restriction to $\mathcal{S}^{(k)}$;

- if one intersects V_k and the other does not, the former is the lower one;

- if no of the two facets intersects V_k, then let $F'^{(k+1)}$, (resp. $G'^{(k+1)}$) be the facet in the boundary of F^k (resp. G^k) which is minimum with respect to $\lhd$. Then

$$F^k \lhd G^k$$

iff either

$$F'^{(k+1)} \lhd G'^{(k+1)}$$

or

$$F'^{(k+1)} = G'^{(k+1)} \quad but \quad G_0^{(k)} \lhd F_0^{(k)}$$

where $F_0^{(k)}$ *(resp.* $G_0^{(k)}$*) means (as above) the unique facet with the same support which intersects* V_k*.*

Moreover, each F^k *intersecting* V_k *is lower than any codimensional-*$(k+1)$ *facet. If* F^k *does not intersect* V_k *then* F^k *is bigger than its minimal boundary* $F'^{(k+1)}$ *and lower than any codimensional -*$(k+1)$ *facet which is bigger than* $F'^{(k+1)}$*.*

This determines the polar ordering of all facets of codimension $\geq k$*.*

□

Remark 4.6 *We ask whether it is possible to characterize polar orderings in purely combinatorial ways. The problem is more or less that of finding a good "combinatorial" description of a flag of subspaces (or better, of half-subspaces) which corresponds to a generic system of polar coordinates, so that we are able to decide what facets belong to coordinate half-spaces. It seems quite reasonable that this can be done by specifying combinatorially a "generic" flag in the given oriented matroid.*

4.4 Combinatorial vector fields

We consider here the regular CW-complex $\mathbf{S} = \mathbf{S}(\mathcal{A})$ of section 3.1. Recall that $k-$cells correspond to pairs $[C \prec F^k]$, where C is a chamber and F^k is a codimensional-k facet in $\mathcal{S}$. We will define a combinatorial gradient vector field Φ over $\mathbf{S}$. One can describe Φ (see section 2.2) as a collection of pairs of cells

$$\Phi \ = \ \{(e,f) \in \mathbf{S} \times \mathbf{S} \mid dim(f) = dim(e) + 1, \ e \in \partial(f)\}$$

so that Φ decomposes into its dimensional-p components

$$\Phi \ = \ \bigsqcup_{p=1}^{n} \Phi_p, \qquad \Phi_p \subset \mathbf{S}_{p-1} \times \mathbf{S}_p$$

($\mathbf{S}_p$ being the $p-$skeleton of $\mathbf{S}$). Let us indicate by

$$\underline{\epsilon} \ , \ \overline{\epsilon} : \ \Phi \ \rightarrow \ \mathbf{S}, \quad \underline{\epsilon}(a,b) = a, \ \overline{\epsilon}(a,b) = b$$

the first and last cells of the pairs of Φ.

We give the following recursive definition:

Definition 4.7 (Polar Gradient) *We define a combinatorial gradient field* Φ *over* $\boldsymbol{S}$ *in the following way:*

the $(j+1)-th$ *component* Φ_{j+1} *of* Φ*,* $j = 0, ..., n-1,$ *is given by all pairs*

$$([C \prec F^j], [C \prec F^{j+1}]), \quad F^j \prec F^{j+1}$$

(same chamber C*) such that*

1. $F^{j+1} \lhd F^j$
2. $\forall F^{j-1} \prec F^j$ *the pair*

$$([C \prec F^{j-1}], [C \prec F^j]) \notin \ \Phi_j$$

Notice that condition (ii) in 3.1 is automatically verified for pairs as in definition 4.7. Condition 2 of 4.7 is empty for the 1-dimensional part Φ_1 of Φ, so

$$\Phi_1 = \{([C \prec C], [C \prec F^1]) \ : \quad F^1 \lhd C\}.$$

According to the definition of generic polar coordinates, only the base-chamber C_0 intersects the origin $O = V_0$, so by theorem 4 all $0-$cells $[C \prec C],\ C \neq C_0$, belong to exactly one pair of Φ_1.

Theorem 6 *One has:*
(i) Φ is a combinatorial vector field on $\boldsymbol{S}$ which is the gradient of a combinatorial Morse function (according to part 2.2).
(ii) The pair

$$([C \prec F^j], [C \prec F^{j+1}]), \quad F^j \prec F^{j+1}$$

belongs to Φ iff the following conditions hold:
(a) $F^{j+1} \lhd F^j$
(b) $\forall\ F^{j-1}$ such that $C \prec F^{j-1} \prec F^j$, one has $F^{j-1} \lhd F^j$.
(iii) Given $F^j \in \mathcal{S}$, there exists a chamber C such that the cell $[C \prec F^j] \in \bar{\epsilon}(\Phi)$ iff there exists $F^{j-1} \prec F^j$ with $F^j \lhd F^{j-1}$. More precisely, for each chamber C such that there exists F^{j-1} with

$$C \prec F^{j-1} \prec F^j, \quad F^j \lhd F^{j-1} \qquad (*)$$

the pair $([C \prec \bar{F}^{j-1}], [C \prec F^j]) \in \Phi$, where $\bar{F}^{j-1}$ is the maximum $(j-1)-$facet (with respect to polar ordering) satisfying conditions (*).
(iv) The set of $k-$dimensional singular cells is given by

$$Sing_k(\boldsymbol{S}) \ = \ \{[C \prec F^k] \ : \ F^k \cap V_k \neq \emptyset,\ F^j \lhd F^k,\ \forall\ C \prec F^j \precneqq F^k\} \quad (13).$$

Equivalently, $F^k \cap V_k$ is the maximum (in polar ordering) among all facets of $C \cap V_k$.

Proof. Clearly Φ_1 satisfies (ii) with $j = 0$. We assume by induction that Φ_j is a combinatorial vector field satisfying (ii). Consider now a $j-$cell $[C \prec F^j] \in \mathbf{S}$. Assume condition (b) of (ii) holds for F^j : then if there exists F^{j+1} with $F^j \prec F^{j+1},\ F^{j+1} \lhd F^j$ (and this happens by theorem 4 iff $F^j \cap V_j = \emptyset$) then

$$([C \prec F^j], [C \prec F^{j+1}]) \in \Phi_{j+1}.$$

If (b) of (ii) does not hold ($j \geq 2$) then let F^{j-1} be the biggest (according to polar ordering) codimensional $j-1$ facet such that

$$C \prec F^{j-1} \prec F^j, \quad F^j \lhd F^{j-1}.$$

Take any F^{j-2} such that $C \prec F^{j-2} \prec F^{j-1}$. We assert that $F^{j-2} \lhd F^{j-1}$. Otherwise, certainly there exists another facet G^{j-1} with

$$F^{j-2} \prec G^{j-1} \prec F^j$$

and by theorem 4 it should be $F^{j-2} \lhd G^{j-1}$, contradicting the maximality of F^{j-1}. So by induction

$$([C \prec F^{j-1}], [C \prec F^j]) \ \in \ \Phi_j$$

(this proves (iii)) and the cell $[C \prec F^j]$ cannot be the origin of a pair of Φ_{j+1}.

To show that Φ_{j+1} is a vector field, we have to see that no cell $[C \prec F^{j+1}]$ is the end of two different pairs of Φ_{j+1}. After $\epsilon-$deforming we reduce to the case where F^{j+1} is $0-$dimensional. Then the unicity of a $j-$facet F^j such that $C \prec F^j \prec F^{j+1}$, and such that (a) and (b) of (ii) hold easily comes from convexity of the chamber C.

This proves both that Φ is a combinatorial vector field and (ii).

Next, we prove that Φ is a gradient field by using theorem 1 of section 3: we have to show that Φ has no closed loops.

So let

$$(\ [C_1 \prec F_1^j], \ [C_1 \prec F_1^{j+1}], \ [C_2 \prec F_2^j], \ [C_2 \prec F_2^{j+1}], \ ...$$

$$..., \ [C_m \prec F_m^j], \ [C_m \prec F_m^{j+1}], \ [C_{m+1} \prec F_{m+1}^j] \ )$$

be a $\Phi-$path (see (1)). First, notice that the $j+1$-facets are ordered

$$F_m^{j+1} \unlhd ... \unlhd F_1^{j+1}.$$

In fact, by definition of path and the boundary in **S** (see sec. 4.1) we have at the k-th step:

$$F_{k+1}^j \prec F_k^{j+1}, \quad F_{k+1}^j \prec F_{k+1}^{j+1}, \quad F_{k+1}^{j+1} \lhd F_{k+1}^j.$$

If also

$$F_k^{j+1} \lhd F_{k+1}^j$$

then by theorem 4 $F_{k+1}^{j+1} = F_k^{j+1}$; otherwise we have necessarily $F_{k+1}^{j+1} \lhd F_{k+1}^j \lhd F_k^{j+1}$. Then if the path is closed it derives (still by theorem 4) that all the F_k^{j+1} equal a unique F^{j+1}. Moreover, up to ϵ-deforming, we can assume that the path is contained in some $V_i(\theta)$ with F^{j+1} a $0-$dimensional facet. Under these assumptions, we show that

$$F_1^j \lhd ... \lhd F_m^j.$$

Let $V_{i-1}(\theta_{i-1}, \theta_i, ...) \subset V_i(\theta)$ be the subspace containing the point F^{j+1}; after ϵ-deforming, the path can be seen inside the subspace

$$\tilde{V} := V_{i-1}(\theta_{i-1} - \epsilon, \theta_i, ...)$$

where for each cell $[C_k \prec F_k^j]$ one has that C_k is a convex open polyhedron in $\tilde{V}$ (may be infinite) and F_k^j is, by point (iii), its *maximum* vertex: all the facets of C_k are lower (in polar ordering) than F_k^j.

By the definition of boundary in 4.1 the two chambers $C_k, \ C_{k+1}$ belong to the same chamber of $\mathcal{A}_{F_{k+1}^j}$. Such a chamber is a convex cone with maximum facet (with respect to polar ordering) is F_{k+1}^j, and such that each of its facets has the same support as some facet of C_{k+1} of the same dimension, having the vertex F_{k+1}^j as one of its 0-facets. Then clearly all the facets of C_k are lower (in

polar ordering) than F^j_{k+1}. In particular $F^j_k \lhd F^j_{k+1}$, which proves that there are no non-trivial closed Φ-paths.

It remains to prove part (iv). In view of (ii), (iii), a cell $[C \prec F^k]$ does not belong to Φ iff

$$F^k \lhd F^{k+1}, \quad \forall\, F^k \prec F^{k+1} \tag{A}$$

and

$$F^{k-1} \lhd F^k, \quad \forall\, C \prec F^{k-1} \prec F^k. \tag{B}$$

Condition (A) holds by theorem 4 iff $P := F^k \cap V_k \neq \emptyset$. Then P is a 0-dimensional facet in V_k, and (B) holds iff P is the maximum facet of the chamber $C \cap V_k$ (according to polar ordering). This is equivalent to (iv), and finishes the proof of the theorem. □

As an immediate corollary we have

Corollary 4.8 *Once a polar ordering is assigned, the set of singular cells is described only in terms of it by*

$$\begin{array}{l} Sing_k(\boldsymbol{S}) \ := \ \{[C \prec F^k] \ : \\ \begin{array}{llllll} a) & F^k \lhd F^{k+1}, & \forall & F^{k+1} & s.t. & F^k \prec F^{k+1} \\ b) & F^{k-1} \lhd F^k, & \forall & F^{k-1} & s.t. & C \prec F^{k-1} \prec F^k \ \} \end{array} \end{array}$$

□

Remark 4.9 *Of course, condition b) of corollary 4.8 is equivalent to:*

$$F' \ \lhd \ F^k \ \text{ for all } F' \text{ in the interval } C \prec F' \ \prec \ F^k.$$

Remark 4.10 *By (iv) of theorem 6 $Sing_k(\boldsymbol{S})$ corresponds to the pairs (C, v) where C is a chamber of the arrangement $\mathcal{A}_k := \mathcal{A} \cap V_k$ and v is the maximum vertex of C. Then v is the minimum vertex of the chamber $\tilde{C}$ of $\mathcal{A}_k$ which is opposite to C with respect to v. Of course, $\tilde{C} \cap V_{k-1} = \emptyset$, so we re-find the one-to-one correspondence between the singular k-cells of $\boldsymbol{S}$ and the chambers of $\mathcal{A}_k$ which does not intersect V_{k-1} (see [Yo05]).*

Remark 4.11 *By easy computation, the integral boundary of the Morse complex generated by singular cells (see [Fo98]) is zero, so we obtain the minimality of the complement. Alternatively, the same result is obtained by noticing that singular cells are in one-to-one correspondence with the set of all the chambers of $\mathcal{S}$ by remark 4.10. But $\sum b_i = |\{chambers\}|$ (see for ex. [Za75, OT92]).*

Remark 4.12 *Our description gives also an explicit additive basis for the homology and for the cohomology in terms of the singular cells in $\boldsymbol{S}$. We can call it a* polar basis *(relative to a given system of generic polar coordinates). It would be interesting to compare such basis with the well-known nbc−basis of the cohomology (see [BZ92, OT92]).*

5 Morse complex for local homology

The gradient field indicates how to obtain a minimal complex from $\mathbf{S}$, by contracting all pairs of cells in the field. For each pair of cells (e^{k-1}, e^k) in Φ, one has a contraction of e^k into $\partial(e^k) \setminus int(e^{k-1})$, by "pushing" $int(e^{k-1}) \cup int(e^k)$ onto the boundary.

In particular, it is possible to obtain a *Morse complex* which computes homology and cohomology, even with local coefficients. We describe here such an algebraic complex, computing homology with local coefficients for the complement $M(\mathcal{A})$. The boundary operators depend only on the partial ordering $\prec$ and on the polar ordering $\lhd$.

First, we give to the coordinate space V_i the orientation induced by the ordered basis $\mathbf{e}_1, ..., \mathbf{e}_i$. Given a codimensional$-i$ facet $F^i \in \mathcal{S}$, the support $|F^i|$ is transverse to V_i, so we give the orthogonal space $|F^i|^\perp$ the orientation induced by that of V_i. Recall from [Sal87] that the complex $\mathbf{S}$ has a real projection $\Re : \mathbf{S} \to \mathbb{R}^n$ which induces a dimension-preserving cellular map onto the *dual* cellularization $\mathcal{S}^\vee \subset \mathbb{R}^n$ of $\mathcal{S}$. We give to a cell $e(F^i) \in \mathcal{S}^\vee$, dual to F^i, the orientation induced by that of $|F^i|^\perp$. We give to a cell $[C \prec F^i] \in \mathbf{S}$ the orientation such that the real projection $\Re : [C \prec F^i] \to e(F^i)$ is orientation preserving.

Let L be a *local system* over $M(\mathcal{A})$, i.e. a module over the group-algebra of the fundamental group $\pi_1(M(\mathcal{A}))$ The basepoint is the origin $O \in C_0$ of the coordinates, which can be taken as the unique $0-$cell of $\mathbf{S}$ (and of $\mathcal{S}^\vee$) contained in C_0. Up to homotopy, we can consider only *combinatorial* paths in the $1-$skeleton of $\mathbf{S}$, i.e. sequences of consecutive edges. Sequences, or *galleries*,

$$C_1, ..., C_t$$

of *adjacent* chambers uniquely correspond to a special kind of combinatorial paths in the $1-$skeleton of $\mathbf{S}$, which we call *positive* paths. Two galleries with the same ends and of minimal length determine two homotopic positive paths (see [Sal87]). One says that a positive path, or gallery, *crosses* an hyperplane $H \in \mathcal{A}$ if two consecutive chambers in the path are separated by H.

Remark that the $1-$dimensional part Φ_1 of the polar field gives a maximal *tree* in the $1-$skeleton of $\mathbf{S}$. Each $0-$cell $v(C)$ of $\mathbf{S}$ is determined by its dual chamber $C \in \mathcal{S}$. Then each $v(C) \in \mathbf{S}$ is connected to the origin O by a unique path $\Gamma(C)$, which is a positive path, determined by a gallery of chambers starting in C and ending in C_0. We have

Lemma 5.1 *For all chambers C, the path $\Gamma(C)$ is minimal, i. e. it crosses each hyperplane at most once.*

Proof. One has that $\Gamma(C)$ consists of a sequence of $1-$cells $[C \prec F^1]$ where $F^1 \lhd C$. It is sufficient to see that the hyperplane $H = |F^1|$ separates C from C_0. This comes immediately from the definition of polar ordering, since one has $P(F^1) = P(C)$ and F^1 is encountered before C by a half-line $V_1(\theta_1, ...)$. $\square$

Notation 5.2 *i) Given two chambers C, C' we denote by $\mathcal{H}(C, C')$ the set of hyperplanes separating C from C'.*

ii) Given an ordered sequence of (possibly not adjacent) chambers $C_1, ..., C_t$ we denote by $u(C_1, ..., C_t)$ the rel-homotopy class of

$$u(C_1, ..., C_t) \;=\; u(C_1, C_2)u(C_2, C_3) \cdots u(C_{t-1}, C_t$$

where $u(C_i, C_{i+1})$ is a minimal positive path induced by a minimal gallery starting in C_i and ending in C_{i+1}. We denote by

$$\bar{u}(C_1, ..., C_t) \in \pi_1(M(\mathcal{A}), O)$$

the homotopy class of a path which is the composition

$$\bar{u}(C_1, ..., C_t) := (\Gamma(C_1))^{-1} u(C_1, ..., C_t)\Gamma(C_t).$$

We denote by

$$\bar{u}(C_1, ..., C_t)_* \;\in\; Aut(L)$$

the automorphism induced by $\bar{u}(C_1, ..., C_t)$.

We need also some definitions.

Definition 5.3 *A cell $[C \prec F] \in \boldsymbol{S}$ will be called* locally critical *if F is the maximum, with respect to $\lhd$, of all facets in the interval $\{F' \;:\; C \prec F' \prec F\}$ of the poset $(\mathcal{S}, \prec)$.*

By corollary 4.8 and remark 4.9 a critical cell is also locally critical. By theorem 6, part (iii), the cell $[C \prec F^k]$ belongs to the $k-$dimensional part Φ_k of the polar field iff it is not locally critical.

Definition 5.4 *Given a codimensional-k facet F^k such that $F^k \cap V_k \neq \emptyset$, a sequence of pairwise different codimensional-$(k-1)$ facets*

$$\mathcal{F}(F^k) \;:=\; (F_{i_1}^{(k-1)}, \cdots, F_{i_m}^{(k-1)}),\; m \geq 1$$

such that

$$F_{i_j}^{(k-1)} \prec F^k,\; \forall\, j$$

and

$$F^k \lhd F_{i_j}^{(k-1)} \; for\; j < m$$

while for the last element

$$F_{i_m}^{(k-1)} \lhd F^k$$

is called an admissible $k-$sequence.

It is called an ordered *admissible $k-$sequence if*

$$F_{i_1}^{(k-1)} \lhd \cdots \lhd F_{i_{m-1}}^{(k-1)}.$$

Notice that in an admissible $k-$sequence with $m = 1$, it remains only a codimensional-$(k-1)$ facet which is lower (in polar ordering) than the given codimensional-k facet.

Two admissible $k-$sequences

$$\mathcal{F}(F^k) \ := \ (F^{(k-1)}_{i_1}, \cdots, F^{(k-1)}_{i_m})$$
$$\mathcal{F}(F'^k) \ := \ (F'^{(k-1)}_{j_1}, \cdots, F'^{(k-1)}_{j_l})$$

$F^k \neq F'^k$, can be *composed* into a sequence

$$\mathcal{F}(F^k)\mathcal{F}(F'^k) \ := \ (F^{(k-1)}_{i_1}, \cdots, F^{(k-1)}_{i_m}, F'^{(k-1)}_{j_1}, \cdots, F'^{(k-1)}_{j_l})$$

when for the last element of the first one it holds

$$F^{(k-1)}_{i_m} \prec F'^k.$$

In case $F^{(k-1)}_{i_m} = F'^{(k-1)}_{j_1}$ we write this facet only once, so there are no repetitions in the composed sequence.

Definition 5.5 *Given a critical $k-$cell $[C \prec F^k] \in \boldsymbol{S}$ and a critical $(k-1)-$cell $[D \prec G^{k-1}] \in \boldsymbol{S}$, an* admissible sequence

$$\mathcal{F} \ = \ \mathcal{F}_{([C \prec F^k], \ [D \prec G^{(k-1)}])}$$

for the given pair of critical cells is a sequence of codimensional-$(k-1)$ facets

$$\mathcal{F} \ := \ (F^{(k-1)}_{i_1}, \cdots, F^{(k-1)}_{i_h})$$

obtained as composition of admissible $k-$sequences

$$\mathcal{F}(F^k_{j_1}) \cdots \mathcal{F}(F^k_{j_s})$$

such that:

a) $F^k_{j_1} = F^k$ *(so $F^{k-1}_{i_1} \prec \ F^k$);*
b) $F^{k-1}_{i_h} \ = \ G^{k-1}$ *and the chamber*

$$C.F^{k-1}_{i_1}. \cdots .F^{k-1}_{i_h}$$

(see notation 4.1) equals D;
c) for all $j = 1, \cdots, h$ the $(k-1)-$cell

$$[C.F^{k-1}_{i_1}. \cdots .F^{k-1}_{i_j} \prec F^{k-1}_{i_j}]$$

is locally critical.

We have an ordered *admissible sequence if all the $k-$sequences that compose it are ordered.*

Lemma 5.6 *All admissible sequences are ordered.*

Proof. Let s be an admissible sequence. One has to show that each $k-$sequence composing s is ordered. This follows by definition 5.5, c), and by the definition of polar ordering. □

Denote by

$$\mathcal{S}eq \;=\; \mathcal{S}eq([C \prec F^k], [D \prec G^{(k-1)}]$$

the set of all admissible sequences for the given pair of critical cells. Of course, this is a finite set which is determined only by the orderings $\prec$, $\lhd$. In fact, the "operation" which associates to a chamber C and a facet F the chamber $C.F$ is detected only by the Hasse diagram of the partial ordering $\prec$. The chamber $C.F$ is determined by: $C.F \prec F$ and $C.F$ is connected by the shortest possible path (= sequence of adjacent chambers) in the Hasse diagram of $\prec$.

Given an admissible sequence $s = (F^{k-1}_{i_1}, ..., F^{k-1}_{i_h})$ for the pair of critical cells $[C \prec F^k]$, $[D \prec G^{k-1}]$, we denote (see notation 5.2) by

$$u(s) \;=\; u(C, C.F^{k-1}_{i_1}, \cdots, C.F^{k-1}_{i_1}.\cdots.F^{k-1}_{i_h})$$

and by

$$\bar{u}(s) = \; \bar{u}(C, C.F^{k-1}_{i_1}, \cdots, C.F^{k-1}_{i_1}.\cdots.F^{k-1}_{i_h}).$$

Set also $l(s) := h$ for the length of s and $b(s)$ for the number of $k-$sequences forming s.

Now we have a complex which computes local system homology.

Theorem 7 *The homology groups with local coefficients*

$$H_k(M(\mathcal{A}), L)$$

are computed by the algebraic complex (C_*, ∂_*) *such that:*

in dimension k

$$C_k \;:=\; \oplus\; L.\; e_{[C \prec F^k]},$$

where one has one generator for each singular cell $[C \prec F^k]$ *in* $\boldsymbol{S}$ *of dimension* k*.*

The boundary operator is given by

$$\partial_k(l.e_{[C\prec F^k]}) \;=\; \sum A^{[C\prec F^k]}_{[D\prec G^{k-1}]}(l).\;\; e_{[D\prec G^{k-1}]} \tag{$*$}$$

($l \in L$ *) where the incidence coefficient is given by:*

$$A^{[C\prec F^k]}_{[D\prec G^{k-1}]} \;:=\; \sum_{s \in Seq} \;(-1)^{l(s)-b(s)}\; \bar{u}(s)_* \tag{$**$}$$

Here the sum is over all possible admissible sequences s *for the pair* $[C \prec F^k]$, $[D \prec G^{k-1}]$. □

Proof. The proof follows by the definition of the vector field, from theorem 6 and from the definition of boundary in **S**. In fact, condition c) implies (by (iii) of theorem 6) that the $(k-1)-$cell $[C.F^{k-1}_{i_1}.\cdots.F^{k-1}_{i_j} \prec F^{k-1}_{i_j}]$ does not belong to Φ_{k-1}, so the pair

$$([C.F^{k-1}_{i_1}.\cdots.F^{k-1}_{i_j} \prec F^{k-1}_{i_j}], [C.F^{k-1}_{i_1}.\cdots.F^{k-1}_{i_j} \prec E^k] \;\in\; \Phi$$

for $j < h$. The result is obtained by substituting to

$$[C.F_{i_1}^{k-1}.\cdots.F_{i_j}^{k-1} \prec F_{i_j}^{k-1}]$$

the remaining boundary

$$\partial([C.F_{i_1}^{k-1}.\cdots.F_{i_j}^{k-1} \prec E^k]) \setminus [C.F_{i_1}^{k-1}.\cdots.F_{i_j}^{k-1} \prec F_{i_j}^{k-1}]$$

and keeping into account the given orientations. $\square$

Remark 5.7 *The sign in formula (**) can be expressed in the following way. If* $s = (F_{i_1}^{k-1}, ..., F_{i_h}^{k-1})$ *then set*

$$\alpha := \ \#\{j < h \ : \ F_{i_j}^{k-1} \lhd F_{i_{j+1}}^{k-1}\}$$

and set $\epsilon = 0$ *or* 1 *according whether the first element* $F_{i_1}^{k-1} \lhd F^k$ *or* $F^k \lhd F_{i_1}^{k-1}$ *Then one has*

$$(-1)^{l(s)-m(s)} \ = \ (-1)^{\alpha+\epsilon} \ .$$

Many admissible sequences in the boundary operator cancel, because of the sign rule. We give a very simplified formula in the following.

Definition 5.8 *1) Given a pair of critical cells* $[C \prec F^k]$, $[D \prec G^{k-1}]$, *we say that an admissible sequence*

$$s = (F_{i_1}^{k-1}, ..., F_{i_h}^{k-1}) \in Seq$$

is m-extensible by *the facet* F'^{k-1} *if:*

a) F'^{k-1} *can be inserted into the sequence* s *to form another sequence* s' *of length* $h+1$ *which is still admissible with respect to the same pair of critical cells, and such that*

$$\bar{u}(s)_* \ = \ \bar{u}(s')_*.$$

b) F'^{k-1} *is the minimum (with respect to* $\lhd$*) codimensional-*$(k-1)$ *facet which satisfies a) (then we call* s' *the m-extension of* s *by* F'^{k-1}*);*

c) F'^{k-1} *is the minimum of the facets* $F"^{k-1}$ *in the sequence* s' *such that the sequence* $s" := s' \setminus F"^{k-1}$ *obtained by removing* $F"^{k-1}$ *is still admissible, and*

$$\bar{u}(s")_* \ = \ \bar{u}(s')_* \ = \ \bar{u}(s)_* \ .$$

In other words, s *is not the m-extension of some* $s"$ *by* $F"^{k-1}$, *with* $F"^{k-1} \lhd F'^{k-1}$.

2) We say that an admissible sequence

$$s = (F_{i_1}^{k-1}, ..., F_{i_h}^{k-1}) \in Seq$$

is m-reducible by F'^{k-1} *in* s, *if the sequence* s' *obtained by removing* F'^{k-1} *is m-extensible by* F'^{k-1}.

Set Seq^e and Seq^r be the set of m-extensible, resp. m-reducible (by some codimensional$-(k-1)$ facet), admissible sequences for a given pair of critical cells. By definition

$$Seq^e \ \cap \ Seq^r \ = \emptyset.$$

The following lemma is also clear from the previous definition.

Lemma 5.9 *There is a one-to-one correspondence*

$$Seq^e \ \leftrightarrow \ Seq^r$$

which associates to a sequence s which is m-extensible by F'^{k-1} its extension s' (obtained by adding F'^{k-1}).

Set

$$Seq^0 \ := \ Seq \setminus (Seq^e \cup Seq^r).$$

as the set of non m-extensible and non m-reducible sequences.

Since the sign in formula (**) which is associated to an m-extensible sequence s and to its extension s' is opposite, it follows:

Theorem 8 *The coefficient of the boundary operator in (**) of theorem 7 holds*

$$A^{[C \prec F^k]}_{[D \prec G^{k-1}]} \ := \sum_{s \in Seq^0} \ (-1)^{l(s)-b(s)} \ \bar{u}(s)_*$$

□

The reduction of theorem 8 is strong.

We consider now *abelian* local systems, i.e. modules L such that the action of $\pi_1(M(\mathcal{A}))$ factorizes through $H_1(M(\mathcal{A}))$. Then to each elementary loop γ_H turning around an hyperplane H in the positive sense it is associated an element $t_H \in Aut(L)$, so one has homomorphisms

$$\mathbb{Z}[\pi_1(M(\mathcal{A}))] \ \to \ \mathbb{Z}[H_1(M(\mathcal{A}))] \ \to \ \mathbb{Z}[t_H^{\pm 1}]_{H \in \mathcal{A}} \subset End(L).$$

An abelian local system as that just defined is determined by the system $\mathcal{T} := \{t_H, \ H \in \mathcal{A}\}$, so we denote it by $L(\mathcal{T})$.

Given an admissible sequence $s = (F^{k-1}_{i_1}, ..., F^{k-1}_{i_h})$ relative to the pair $[C \prec F^k]$, $[D \prec \mathcal{G}^{k-1}]$, and given an hyperplane $H \in \mathcal{A}$, we indicate by $\mu(s, H)$ the number of times the path $u(s)$ crosses H.

Lemma 5.10 *For s, H as before, one has*

1) $H \in \mathcal{H}(C_0, C) \cap \mathcal{H}(C_0, D)$ then

$$\begin{array}{ll} \mu(s,H) = 0 \ \textit{if} & F^k \not\subset H \ \textit{or} \ \ F^k \subset H \ \textit{and} \ F^{k-1}_{i_1} \lhd F^k \\ \mu(s,H) = 2 \ \textit{otherwise} & \end{array}$$

2) $H \in \mathcal{H}(C_0, D) \cap \mathcal{H}(C, D)$ then $\mu(s, H) = 1$
3) $H \in \mathcal{H}(C_0, C) \cap \mathcal{H}(C, D)$ then
if $F^k \not\subset H$ then $\mu(s, H) = 1$;

if $F^k \subset H$ then

$$
\begin{array}{ll}
F_{i_1}^{k-1} \lhd F^k \;\Rightarrow & \mu(s,H) = 1 \\
F^k \lhd F_{i_1}^{k-1} \;\Rightarrow & \\
\quad \mu(s,H) = 3 & \textit{if } H \textit{ separates } C_0 \textit{ from the first element in } s \textit{ which is lower} \\
\quad\quad \textit{than } F^k; & \\
\quad \mu(s,H) = 1 & \textit{otherwise.}
\end{array}
$$

If H does not separate any two among $C_0,\ C,\ D \Rightarrow \mu \leq 2$

Proof. The proof is very similar to that of lemma 5.1. $\square$

Theorem 9 *For the local system $L(\mathcal{T})$ the coefficient $\bar{u}(s)_*$ in theorem 8 is given by*

$$\bar{u}(s)_* \;=\; \prod_{H \in \mathcal{A}} t_H^{m(s,H)}$$

where if $s = (F_{i_1}^{k-1}, ..., F_{i_h}^{k-1})$ then

$$m(s,H) \;:=\; \left[\frac{\mu(s,H) - \epsilon(C) + \epsilon(D)}{2} \right]$$

where $\epsilon(C)$ (resp. $\epsilon(D)$) holds 1 or 0 according whether H separetes the base chamber C_0 from C (resp. D).

Therefore one always has $m(s,H) \leq 1$, with $m(s,H) = 1$ if
i) $H \in \mathcal{H}(C_0, C) \cap \mathcal{H}(C, D)$ and

$$F^k \subset H,\ F^k \lhd F_{i_1}^{k-1}$$

with H separating C_0 from the first element in s which is lower than F^k;
ii) $H \in \mathcal{H}(C_0, C) \cap \mathcal{H}(C_, D)$ and

$$F^k \subset H,\ F^k \lhd F_{i_1}^{k-1}.$$

In the other cases we have

$$m(s,H) \leq 1$$

if H does not separates any two of the three chambers $C_0,\ C,\ D$, otherwise

$$m(s,H) = 0.$$

Proof. The proof follows directly from the previous lemma, by computing, for each s, the number of times the path $\bar{u}(s)$ turns around some hyperplane. $\square$

Theorem 9 gives an efficient algorithm to compute abelian local systems in terms of the polar ordering (see also [Co93, CO00, ESV92, Ko86, LY00, Sal94, STV95, Su02, Yo05]).

6 The braid arrangement

In this section, we describe the combinatorial gradient vector field for the *braid arrangement* $\mathcal{A} = \{H_{ij} = \{x_i = x_j\},\ 1 \leq i < j \leq n+1\}$. Let us start with some notations.

6.1 Tableaux description for the complex $\mathbf{S}(A_n)$

We indicate simply by A_n the symmetric group on $n+1$ elements, acting by permutations of the coordinates. Then $\mathcal{A} = \mathcal{A}(A_n)$ is the braid arrangement and $\mathbf{S}(A_n)$ is the associated CW-complex (see 4.1).

Given a system of coordinates in $\mathbb{R}^{n+1}$, we describe $\mathbf{S}(A_n)$ through certain tableaux as follow.

Every k-cell $[C \prec F]$ is represented by a tableau with $n+1$ boxes and $n+1-k$ rows (aligned on the left), filled with all the integers in $\{1, ..., n+1\}$. There is no monotony condition on the lengths of the rows. One has:

- $(x_1, \dots, x_{n+1})$ is a point in F iff:

1. i and j belong to the same row iff $x_i = x_j$,
2. i belongs to a row less than the one containing j iff $x_i < x_j$;

- the chamber C belongs to the half-space $x_i < x_j$ iff:

1. either the row which contains i is less than the one containing j or
2. i and j belong to the same row and the column which contains i is less than the one containing j.

Notice that the geometrical action of A_n on the stratification induces a natural action on the complex $\mathbf{S}$, which, in terms of tableaux, is given by a left action of A_n: $\sigma.\ T$ is the tableau with the same shape as T, and with entries permuted through σ.

6.2 Construction of singular tableaux and polar ordering

In this part we use theorem 5 constructing and ordering "singular" tableaux, corresponding to codimensional-k facets which intersect V_k. We give both an algorithmic construction, generating bigger dimensional tableaux from the lower dimensional ones, and an explicit one.

Denote by $\mathbf{T}(A_n)$ the set of "row-standard" tableaux, i.e. with entries increasing along each row. Each facet in $\mathcal{S}$ corresponds to an equivalence class of tableaux, where the equivalence is up to row preserving permutations. So there is a $1-1$ correspondence between $\mathbf{T}(A_n)$ and the set of facets in $\mathcal{A}(A_n)$. Let $\mathbf{T^k}(A_n)$ be the set of tableaux of dimension k (briefly, $k-$tableaux), i.e. tableaux with exactly $n+1-k$ rows. Moreover, write $T \prec T'$ iff $F \prec F'$, where the tableaux T and T' correspond respectively to F and F'. Our aim is to give a polar ordering on $\mathbf{T}(A_n)$.

Definition 6.1 (Moving Function) *Fixed an integer* $1 \leq r \leq n+1$*, for each* $0 \leq j \leq n-k$*, define the* moving function

$$M_{j,r} : \mathbf{T^{k+1}}(A_n) \longrightarrow \mathbf{T}(A_n),$$

where the tableau $M_{j,r}(T^{k+1})$ is obtained from T^{k+1} moving the entry r to the j-th row. Case $j = 0$ means that r becomes the only entry of the first row in $M_{j,r}(T^{k+1})$.

Of course, if r is the unique element of its row, moving r makes the preceding and following rows to become adjacent. So, the number of rows of the new tableau can increase or decrease by 1, or it can remain equal (when the row of r has at least two elements and $j > 0$). Given a tableau T^k, where r is in the i-th row, we define the set of tableaux $M_r(T^k) = \{M_{j,r}(T^k)\}_{0<j<i}$. We assign to $M_r(T^k)$ the reverse order with respect to j.

Let us consider the natural projection $p_{n,m} : \mathbf{T}(A_n) \longrightarrow \mathbf{T}(A_m)$ obtained by forgetting the entries $r \geq m+2$ in each tableau ("empty" rows are deleted). For any $T \in \mathbf{T}(A_n)$ denote by m_T the minimum integer $1 \leq m \leq n$ such that $p_{n,m}(T)$ preserves the dimension of T. So, each $j > m_T+1$ is the unique element of its row.

Definition 6.2 (T-Blocks) *Let T be a k-tableau in $\mathbf{T}(A_n)$ and $e_i(T)$ the first entry of its i-th row. Then, if $m_T < n$, for any integer $m_T+1-k < h \leq n+1-k$ we define a new ordered set*

$$\mathcal{Q}_{n,h}(T) = \bigcup_{m_T+1-k<i\leq h} M_{e_i(T)}(T_{i-1}), \;\; T_{m_T-k+1} = T \text{ and } T_i = M_{0,e_i(T)}(T_{i-1}), \tag{2}$$

where $M_{e_i(T)}(T_{i-1})$ are already ordered and tableaux in $M_{e_i(T)}(T_{i-1})$ are less than tableaux in $M_{e_j(T)}(T_{j-1})$ iff $i < j$.

Let $T \in \mathbf{T}(A_n)$ be a tableau representing a facet F. The symmetry in $\mathbb{R}^{n+1}$ with respect to the affine subspace generated by F preserves the arrangement, so it induces an involution r_T on $\mathbf{T}(A_n)$. Given a k-tableau $T \in \mathbf{T^k}(A_n)$ with $m_T < n$, let $\mathcal{Q}_{n,h}(T) = \{T_i\}_{1\leq i\leq p}$, where the indices follow the ordering introduced in the previous definition. If we consider a k-tableau $\overline{T}$ then we can define recursively $\overline{T}_i$ as follow:

1. $\overline{T}_1 = \overline{T}$
2. $\overline{T}_i = r_{T_i}\overline{T}_{i-1}$ if $T_i \succ \overline{T}_{i-1}$, $\overline{T}_i = \overline{T}_{i-1}$ otherwise.

Denote the last tableau $\overline{T}_p$ by $r_{\mathcal{Q}_{n,h}(T)}(\overline{T})$.

Let $i_{m,n} : \mathbf{T}(A_m) \longrightarrow \mathbf{T}(A_n)$ be the natural inclusion map, i.e. $i_{m,n}(T)$ is obtained by attaching to T exactly $n-m$ rows of lenght one having entries $m+2, \dots, n+1$ increasing along the first column.

Let $\pi_0(A_n)$ be the set given by the identity 0-tableau (i.e., one column with growing entries); we define $\pi_{k+1}(A_n) \subset \mathbf{T^{k+1}}(A_n)$ as the image of the map:

$$\begin{aligned} \overline{\mathcal{Q}}_{n,n+1-k} :& \pi_k(A_{n-1}) \longrightarrow \mathbf{T^{k+1}}(A_n), \\ & T_i \longrightarrow \mathcal{Q}_{n,n+1-k}(T_{i,i}) \end{aligned} \tag{3}$$

where $T_{i,1} = i_{n-1,n}(T_i)$ and $T_{i,j} = r_{\mathcal{Q}_{n,n+1-k}(T_{j-1,j-1})}(T_{i,j-1})$ for $j \leq i$.

We inductively order $\pi_k(A_n)$ by requiring that the map in 3 is order preserving and using the ordering of the T-blocks involved.

Remark 6.3 *Remark that the k-tableau $T_{i,i}$ in the above definition is $T_{i,i} = r_T(T_{i,1})$ where $T = i_{m_{T_i},n}(T^{m_{T_i}})$ and $T^{m_{T_i}}$ is the unique tableau (having only one row) of $\mathbf{T^{m_{T_i}}}(A_{m_{T_i}})$.*

Now let us describe directly tableaux T^k in $\pi_k(A_n)$. Define the following operations between tableaux:

1. $T * T'$ is the new tableau obtained by attaching vertically T' below T.
2. $T *_i h$ is the tableau obtained by attaching the one-box tableau with entry h to the i-th row of T.
3. T^{op} is the tableau obtained from T by reversing the row order. Notice that $(T * T')^{op} = T'^{op} * T^{op}$.

Let us fix k integers $1 < j_1 < \cdots < j_k \leq n+1$ and, for $1 \leq h \leq k+1$, let T_h be the 0-tableau (= one-column tableau) with entries $J_h = \{j_{h-1}+1, \ldots, j_h - 1\}$ in the natural order (set $j_0 = 0,\ j_{k+1} = k+2$).
Then, for any suitable choice of integers $i_1, \ldots, i_k$ we define a k-tableau:

$$T^k = ((\cdots((((T_1^{op} *_{i_1} j_1) * T_2)^{op} *_{i_2} j_2) * T_3)^{op} \cdots)^{op} *_{i_k} j_k) * T_{k+1}. \qquad (4)$$

Proposition 6.4 *A k-tableau in $\mathbf{T}(A_n)$ is in $\pi_k(A_n)$ iff it is of the form (4). Moreover, the order in $\pi_k(A_n)$ is the one induced by lexicographic order between sequences of pairs $((j_1, i_1), \ldots, (j_k, i_k))$, where $(j_t, i_t) < (j'_t, i'_t)$ iff either $j_t < j'_t$ or $j_t = j'_t$ and $i_t > i'_t$.*

Proof. The proof is by induction on the dimension n of $\mathcal{A}(A_n)$.
The result holds trivially for $n = 1$.
Let T^k be a tableau in $\mathbf{T}(A_n)$ such that the $(n+1-k)$-th row has length one and entry $n+1$. Then, by construction, $T^k \in \pi_k(A_n)$ iff $p_{n,n-1}(T^k) \in \pi_k(A_{n-1})$ and proof comes by inductive hypothesis.
Otherwise $j_k = n+1$, i.e.

$$T^k = ((\cdots(((T_1^{op} *_{i_1} j_1) * T_2)^{op} *_{i_2} j_2 * T_3)^{op} \cdots)^{op} *_{i_{k-1}} j_{k-1} * T_k)^{op} *_{i_k} (n+1).$$

If we define a $(k-1)$-tableau as

$$T^{k-1} = (\cdots(((T_1^{op} *_{i_1} j_1) * T_2)^{op} *_{i_2} j_2 * T_3)^{op} \cdots)^{op} *_{i_{k-1}} j_{k-1} * T_k$$

then $T^{k-1} \in \pi_{k-1}(A_{n-1})$ by induction and $T^k \in \overline{\mathcal{Q}}_{n,n+1-(k-1)}(T^{k-1})$ by construction, i.e. $T^k \in \pi_k(A_n)$. Since, by construction, given T, T' belonging to $\overline{\mathcal{Q}}_{n,n+1-(k-1)}(T^{k-1})$, one has that T is lower than T' iff either $j_k < j'_k$ or $j_k = j'_k$ and $i_k > i'_k$ then the proof arises from inductive hypothesis. $\square$

Let us consider the subset $\mathcal{U}^k(A_n)$ of rank-k elements in the lattice $L(\mathcal{A}(A_n))$ (see [OT92]): in other words, the set of codimensional-k intersections of hyperplanes from $\mathcal{A}$. The support of the facet represented by $T^k \in \mathbf{T^k}(A_n)$ is denoted by $\mid T^k \mid \in \mathcal{U}^k(A_n)$.

By arguments similar to those used in the proof of proposition 6.4, one obtains the following result.

Lemma 6.5 *$\pi_k(A_n)$ is a complete system of representatives for $\mathcal{U}^k(A_n)$, i.e. any affine space in $\mathcal{U}^k(A_n)$ is the support of a tableau in $\pi_k(A_n)$ and any two k-tableaux in $\pi_k(A_n)$ have different supports.* $\square$

Remark 6.6 *It follows that the cardinality of $\pi_k(A_n)$ is the number of k-codimensional subspaces of the intersection lattice $L(\mathcal{A}(A_n))$, i.e. the Stirling number $S(n+1, n+1-k)$ (see [OT92]).*

Now let us prove that tableaux in $\pi_k(A_n)$ describe critical cells of $\mathbf{S}^k(A_n)$ with respect to a suitable system of polar coordinates.

Proposition 6.7 *There exists a system of polar coordinates, generic with respect to $\mathcal{A}(A_n)$, such that a codimensional-k facet F meets the V_k space iff the tableau which represent F is in $\pi_k(A_n)$. Moreover, the induced order between codimensional-k facets intersecting V_k equals that introduced before for $\pi_k(A_n)$.*

Proof. We start defining $\mathcal{A}(A^n_{n-1}) = i_{n-1,n}(\mathcal{A}(A_{n-1}))$, $\mathcal{A}(A^n_{n-1})^c = \mathcal{A}(A_n) \setminus \mathcal{A}(A^n_{n-1})$. Let also $\pi_{k-1}(A^n_{n-1}) = i_{n,n-1}(\pi_{k-1}(A_{n-1}))$.

The proof is by double induction on the dimension n of $\mathcal{A}(A_n)$ and the dimension k of sections V_k. The result holds trivially for $n = 1,\ 2$ and also for $k = 0$ and any n.

By induction, it is possible to find a system of generic polar coordinates $V'_0, ..., V'_n$ in $\mathbb{R}^n$ which verifies the theorem for A_{n-1}. By using arguments similar to those used in section 4.2 one can embed this system to a generic one $V_0, ..., V_n,\ V_{n+1} = \mathbb{R}^{n+1}$ for A_n, where the embedding is compatible with $i_{n,n-1}$ (i.e., it takes $\mathcal{A}(A_{n-1})$ inside $\mathcal{A}(A^n_{n-1})$).

By induction on k, we assume that the system verifies the assertion until codimensional-$(k-1)$ facets.

Let $\mathcal{L}_k(\pi_{k-1}(A^n_{n-1}))$ be the set of all affine lines realized as intersections between V_k and $\mathcal{U}^{k-1}(\mathcal{A}(A^n_{n-1}))$. By lemma 6.5 any line L_i in $\mathcal{L}_k(\pi_{k-1}(A^n_{n-1}))$ lies in the support of one and only one tableau $T^{k-1}_i \in \pi_{k-1}(A^n_{n-1})$.

Now notice that for any $T^{k-1}_i \in \pi_{k-1}(A^n_{n-1})$, the last row is composed only by the entry $(n+1)$. Moreover, by remark 6.3, the tableau $T^{k-1}_{i,i}$ is obtained from T^{k-1}_i without moving the entry $n+1$. Then, by construction, $\pi_k(A^n_{n-1})$ is given by the ordered union of $\mathcal{Q}_{n,n-(k-1)}(T^{k-1}_i)$ for $T^{k-1}_i \in \pi_{k-1}(A^n_{n-1})$.

Therefore (by induction) the line L_i intersects in order all k-facets represented by $\mathcal{Q}_{n,n-(k-1)}(T^{k-1}_i)$ and, after that, all hyperplanes in $\mathcal{A}(A^n_{n-1})^c$. Obviously these last intersections have to be along a gallery of k-tableaux starting from the $(k-1)$-tableau

$$\tilde{T}^{k-1}_i := r_{\mathcal{Q}_{n,n-(k-1)}(T^{k-1}_i)}(T^{k-1}_i).$$

But $M_{e_{(n+1)-(k-1)}(\tilde{T}^{k-1}_i)}(\tilde{T}^{k-1}_i) = M_{n+1}(\tilde{T}^{k-1}_i)$ is the only choice in order to have a gallery throughout hyperplanes in $\mathcal{A}(A^n_{n-1})^c$ and starting from $\tilde{T}^{k-1}_i$.

This proves the first statement of the proposition.

According to definition 4.4 let

$$P_{F^k_{i,h}} \ := \ clos(F^k_{i,h}) \cap V_k$$

where $F^k_{i,h}$ is the facet represented by the tableau $T^k_{i,h} \in \mathcal{Q}_{n,(n+1)-(k-1)}(T^{k-1}_i)$. By definition 4.5 we need to understand the ordering of such points $P's$.

By the above considerations and the inductive hypothesis it follows that in $\pi_k(A^n_{n-1})$ one has

$$P_{F^k_{i_1,h_1}} \lhd P_{F^k_{i_2,h_2}}$$

iff the pair (i_2, h_2) follows the pair (i_1, h_1) according to the lexicographic ordering. By simple geometric considerations this lexicographic ordering is preserved when we pass to $\pi_k(A_n)$. But this corresponds exactly to the ordering which we defined before for $\pi_k(A_n)$. $\square$

Remark 6.8 *By theorem 5, we can reconstruct the ordering of* $\mathbf{T}(A_n)$ *from that of* $\pi_k(A_n),\ k = 0, ...n$.

In order to identify critical cells of $\boldsymbol{S}(\mathcal{A}(A_n))$ *we just apply theorem 6.*

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